Nairian models and forcing axioms

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The Proper Forcing Axiom

Conjecture

PFA implies there is an inner model with a supercompact cardinal.

Theorem (Todorčević)

Assume PFA. Then $\neg\Box(\kappa)$ *for all* $\kappa > \omega_1$*.*

- Lower bound computations for PFA go through failures of square
  - Schimmerling, Steel, Jensen-Schimmerling-Schindler-Steel
  - Sargsyan-Trang have obtained a model of LSA

Conjecture

1. (Zeman) $\neg\Box_\kappa$ for κ singular is equiconsistent with a subcompact
2. (Steel) $\neg\Box_\kappa$ for κ singular strong limit requires a superstrong

The Proper Forcing Axiom

Theorem (Viale-Weiss)

Suppose κ is an inaccessible cardinal and PFA is forced by an iteration $\mathbb{P}$ collapsing κ to ω_2 such that

1. *$\mathbb{P}$ is the direct limit of an iteration $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$ which takes direct limits stationarily often, and*
2. *$|\mathbb{P}_\alpha| < \kappa$ for all $\alpha < \kappa$.*

Then κ is strongly compact. If $\mathbb{P}$ is proper, then κ is supercompact.

The Proper Forcing Axiom

- I.e. If PFA holds and N is an inner model with the ω_2-cover and ω_2-approximation properties in which ω_2 is inaccessible, then ω_2^V is strongly compact.
- (Usuba) If δ is weakly compact and $V[G]$ is a δ-cc forcing extension, then V has the δ-cover and δ-approximation properties in $V[G]$.

Key question

Are there any other methods for building models of PFA/MM?

Woodin's consistency proof for Martin's Maximum

Theorem (Woodin)

Assume there is a Vopěnka cardinal δ *and there is an elementary embedding* $j : V_\delta \to V_\delta$ *with* $V_\kappa \prec V_\delta$, *where* $\kappa = \mathrm{crit}(j)$. *Then there is a revised countable support iteration* $\mathbb{P}$ *of semiproper forcings such that if* $g \subseteq \mathbb{P}$ *is* V*-generic, then*

$$V[g]_\delta \models \mathrm{ZFC} + \mathrm{MM}^{++}.$$

Moreover, in $V[g]_\delta$ *there is no proper inner model of ZFC with the* $\omega_2^{V[g]}$*-cover and* $\omega_2^{V[g]}$*-approximation properties in* $V[g]_\delta$.

- Vacuous if the HOD conjecture is true

Axiom $(*)^{++}$

Theorem (Aspero-Schindler)

Assume MM^{++}*. Then axiom* $(*)$ *holds.*

Definition (Axiom $(*)^{++}$)

There is a pointclass $\Gamma \subset \wp(\mathbb{R})$ and $g \subseteq \mathbb{P}_{\max}$ such that

1. $L(\Gamma, \mathbb{R}) \models \mathrm{AD}^+$,
2. g is $L(\Gamma, \mathbb{R})$-generic, and
3. $\wp(\mathbb{R}) \in L(\Gamma, \mathbb{R})[g]$.

Question (Woodin)

Is MM^{++} consistent with $(*)^{++}$? Is SRP consistent with $(*)^{++}$?

The cofinality of $\Theta^{L(\Gamma^\infty,\mathbb{R})}$

- Θ is the least ordinal which is not the surjective image of $\mathbb{R}$
- Γ^∞ denotes the collection of universally Baire sets

Theorem (Woodin)

1. $(*)^{++}$ *implies* $\Theta^{L(\Gamma^\infty,\mathbb{R})} = \omega_3$
2. *Suppose* δ *is a supercompact cardinal. If there are class many Woodin cardinals and* $V[g]$ *is a* δ*-cc forcing extension in which* $\delta = \omega_2$*, then* $V[g] \models \Theta^{L(\Gamma^\infty,\mathbb{R})} < \omega_3$.

Forcing over models of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular."

Theorem (Woodin)

Assume $\mathrm{AD}_{\mathbb{R}}$ + *"Θ is regular." Then* $\mathbb{P}_{\max} * Add(\omega_3, 1) \Vdash \mathrm{MM}^{++}(c)$.

Theorem (Caicedo-Larson-Sargsyan-Schindler-Steel-Zeman)

Assume $\mathrm{AD}_{\mathbb{R}}$ + *"Θ is regular". Suppose the set of κ which are regular in HOD and have cofinality ω_1 is stationary in Θ. Then*

$$\mathbb{P}_{\max} * Add(\omega_3, 1) \Vdash \mathrm{MM}^{++}(c) + \neg\Box(\omega_2) + \neg\Box_{\omega_2}.$$

Theorem (Larson-Sargsyan)

Assume $\mathrm{AD}_{\mathbb{R}} + \exists\lambda \bowtie_\lambda$ *. Then*

$$\mathbb{P}_{\max} * Add(\omega_3, 1) * Add(\omega_4, 1) \Vdash \neg\Box(\omega_3) + \neg\Box(\omega_4).$$

- (Woodin) MM^{++} cannot be forced over a determinacy model of the form $L(S, \wp(\mathbb{R}))$ for $S \subset \mathrm{Ord}$.

Nairian models

Definition

Assume LSA and let $(\theta_\gamma : \gamma \leq \Omega)$ be the Solovay sequence. Suppose $\alpha + 1 \leq \Omega$ is such that

$$\mathrm{HOD} \models \text{"}\theta_{\alpha+1} \text{ is a limit of Woodin cardinals."}$$

Let $M = V^{\mathrm{HOD}}_{\theta_{\alpha+1}}$ and $N = L_{\theta_{\alpha+1}}(\bigcup_{\eta<\theta_{\alpha+1}}(M|\eta)^\omega)$.

- (Woodin, building on Steel) $N \models \mathrm{ZF}$.
- For this talk, a *Nairian model* is an initial segment N_γ of N such that $N_\gamma \models \mathrm{ZF}$.
- Nairian models exist assuming less than a Woodin limit of Woodin cardinals

Theorem (B.-Sargsyan)

For $i \in \{1, 2, 3\}$, the theory

1. *there are class many Woodin cardinals,*
2. *Γ^∞ is sealed, and*
3. *$\Theta^{L(\Gamma^\infty,\mathbb{R})} = \omega_i$*

is consistent.

- For ω_2 and ω_3, uses forcing over Nairian models

Failures of square

Theorem (B.-Larson-Sargsyan)

Fix $n < \omega$. In a forcing extension of a Nairian model, $\neg\square(\aleph_i)$ holds for all $i \in [2, n]$.

Theorem (B.-Larson-Sargsyan)

Let N_γ be the least initial segment of N such that $N_\gamma \models \mathrm{ZF}$. Then in a forcing extension $N_\gamma[g]$ of N_γ, $\neg\square(\kappa)$ holds for all $\kappa > \omega_1$.

Corollary

$\mathrm{ZFC} + \forall\kappa > \omega_1 \neg\square(\kappa) <_{\mathrm{Con}} \mathrm{ZFC} + \mathrm{WLW}.$

- (Neeman-Steel) Assuming an iterability hypothesis, if δ is a Woodin cardinal such that $\neg\square_\delta + \neg\square(\delta)$, then there is an inner model of ZFC + "there is a subcompact cardinal."

The HOD conjecture I

Definition

A regular cardinal $\kappa > \omega_1$ is *ω-strongly measurable in* HOD if there is $\gamma < \kappa$ such that

1. $(2^\gamma)^{\mathrm{HOD}} < \kappa$
2. $\{\alpha < \kappa : \mathit{cof}(\alpha) = \omega\}$ cannot be definably partitioned into γ sets.

Theorem (HOD Dichotomy theorem; Woodin, Goldberg)

If δ is a supercompact cardinal, then exactly one of the following hold.

1. *No regular cardinal $\gamma \geq \delta$ is ω-strongly measurable in HOD.*
2. *Class many regular cardinals are ω-strongly measurable in HOD.*

- **HOD Hypothesis**: There is a class of regular cardinals which are not ω-strongly measurable in HOD.

The HOD conjecture II

Definition (HOD conjecture)

ZFC + "there is a supercompact cardinal" $\vdash$ the HOD Hypothesis.

Theorem (Ben Neria-Hayut)

It is consistent relative to an inaccessible cardinal κ with $\kappa = \sup_{\alpha<\kappa} o(\alpha)$ that every successor of a regular cardinal is ω-strongly measurable in HOD.

Theorem (B.-Larson-Sargsyan)

In $N_\gamma[g]$, successors of singular cardinals are ω-strongly measurable in HOD.

- The HOD Hypothesis is not provable in ZFC.

Stationary Set Reflection I

Definition (SRP; Todorcevic)

Suppose $\lambda > \omega_1$ and $S \subset \wp_{\omega_1}(\lambda)$ is projective stationary. Then for every $X \subseteq \lambda$ such that $\omega_1 \subseteq X$ and $|X| = \omega_1$, there is $X \subseteq Y \subseteq \lambda$ of size ω_1 such that $S \cap \wp_{\omega_1}(Y)$ contains a club in $\wp_{\omega_1}(Y)$.

Theorem (Woodin)

Assume SRP. Then exactly one of the following hold.

1. $\mathrm{Ord}^\omega \subset \mathrm{HOD}$.
2. *There is an ordinal α such that every regular cardinal $\kappa > \alpha$ is ω-strongly measurable in* HOD.

Definition (The (HOD+SRP) conjecture)

$\mathrm{ZFC} + \mathrm{SRP}$ proves that $\mathrm{Ord}^\omega \subset \mathrm{HOD}$.

Stationary Set Reflection II

Definition (SRP^*; Woodin)

Suppose $\lambda > \omega_1$. There is a normal fine ideal $I \subset \wp(\wp_{\omega_1}(\lambda))$ such that

1. for every stationary $T \subset \omega_1$, the set $\{\sigma \in \wp_{\omega_1}(\lambda) | \sigma \cap \omega_1 \in T\} \notin I$
2. if $S \subset \wp_{\omega_1}(\lambda)$ is such that for each stationary $T \subset \omega_1$, $\{X \in S : X \cap \omega_1 \in T\} \notin I$, there is $\omega_1 < \gamma < \lambda$ such that $S \cap \wp_{\omega_1}(\gamma)$ contains a club in $\wp_{\omega_1}(\gamma)$.

Theorem (Steel-Zoble)

Suppose NS_{ω_1} is saturated, $2^\omega \leq \omega_2$, and $\mathrm{SRP}^*(\omega_2)$. Then $L(\mathbb{R}) \models \mathrm{AD}$.

Theorem (B.-Sargsyan)

SRP^* *holds in $N_\gamma[g]$.*

Stationary Set Reflection III

Quasi-Club Conjecture

In Nairian models, the quasi-club filter on $\wp_{\omega_1}(\lambda)$ is an ultrafilter for all $\lambda \in \mathrm{Ord}$.

Theorem (B. Sargsyan)

Assuming the Quasi-Club conjecture, $N_\gamma[g] \models \mathrm{SRP}$.

Corollary

The (HOD+SRP) conjecture is false, assuming the Quasi-Club conjecture.

- ω_2 is a supercompact cardinal in N_γ

Theorem (B.-Sargsyan)

SRP^* *is consistent with* $(*)^{++}$. *If the Quasi-Club conjecture is true, then* SRP *is consistent with* $(*)^{++}$.

Consequences of MM

Theorem (B.-Sargsyan)

The following hold in $N_\gamma[g]$.

- *Moore's Open Mapping Reflection, and*
- MA^+(*σ-closed*).

Question

Does the P-Ideal Dichotomy hold in $N_\gamma[g]$?

Question

1. Is MM equivalent to some conjunction of these principles?
2. Could some fragment of MM^{++} suffice to produce a "minimal" model of MM^{++}?

Conclusion: Further Questions

Question

Is ω_1 a Θ^+-Berkeley cardinal in the full Nairian model?[1]

Question

Can a Woodin cardinal exist in a forcing extension of a Nairian model?

Question

Are the universally Baire sets sealed in full Nairian models? In their ZFC forcing extensions?

Question

What is the axiomatic theory of N?

Thank you

[1] A cardinal κ is η-*Berkeley* if for every transitive M with $|M| < \eta$ and every $\alpha < \kappa$, there is an elementary embedding $j : M \to M$ with $\mathrm{crit} \in (\alpha, \kappa)$.